SUBJECT: MATHEMATICS

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Remember, Eigen values and eigen vectors can be calculated of Fredholm integral equation when is homogeneous i.e. $f(x)=0$ and after having λ if not there.

(1) Find eigen values and eigen functions of

$$\phi(x) = \lambda \int_0^{\pi} \{\cos^2 x \cos 2\xi + \cos 3x \cos^3 \xi\} \phi(\xi) d\xi \quad \text{--- (1)}$$

$$\Rightarrow \phi(x) = \lambda \cos^2 x \cdot c_1 + \lambda \cos 3x \cdot c_2 \quad \text{--- (2)}$$

where

$$c_1 = \int_0^{\pi} \cos 2\xi \, \phi(\xi) d\xi \quad \text{--- (3)}$$

$$c_2 = \int_0^{2\pi} \cos^3 \xi \, \phi(\xi) d\xi \quad \text{--- (4)}$$

from (2), $\phi(\xi) = \lambda c_1 \cos^2 \xi + \lambda c_2 \cos 3\xi$ — (5)

Eq^{n}s (3) & (5) $\Rightarrow$

$$c_1 = \int_0^{\pi} \cos 2\xi \left[\lambda c_1 \cos^2 \xi + \lambda c_2 \cos 3\xi\right] d\xi$$

$$= \int_0^{\pi} \left[\cos 2\xi \left[\lambda c_1 \cdot \frac{1+\cos 2\xi}{2}\right] + \lambda c_2 \cos 3\xi \cos 2\xi\right] d\xi$$

$$c_1 = \frac{\lambda c_1}{2} \int_0^{\pi} \cos 2\xi \, d\xi + \frac{\lambda c_1}{2} \int_0^{\pi} \cos^2 2\xi \, d\xi + \lambda c_2 \int_0^{\pi} \cos 3\xi \cos 2\xi \, d\xi$$

$$c_1 = 0 + \frac{\lambda c_1}{2} \int_0^{\pi} \frac{1+\cos 4\xi}{2} d\xi + \frac{\lambda c_2}{2} \int_0^{\pi} (\cos 5\xi + \cos \xi) d\xi$$

$$c_1 = \frac{\lambda c_1}{4} \cdot \pi + \frac{\lambda c_2}{2} \cdot 0 \Rightarrow \left(1 - \frac{\lambda \pi}{4}\right) c_1 + 0 c_2 = 0 \quad \text{--- (6)}$$

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Eqn's (4) and (5) $\Rightarrow$

$$c_2 = \int_0^\pi \cos^3\xi \left[\lambda c_1 \cos^2\xi + \lambda c_2 \cos 3\xi\right] d\xi$$

$$c_2 = \int_0^\pi \left(\frac{\cos 3\xi + 3\cos\xi}{4}\right)\left[\lambda c_1 \left(\frac{1+\cos 2\xi}{2}\right) + \lambda c_2 \cos 3\xi\right] d\xi$$

$$\Rightarrow c_2 = \frac{\lambda c_1}{8}\int_0^\pi (\cos 3\xi + 3\cos\xi)(1+\cos 2\xi)\, d\xi$$

$$+ \frac{\lambda c_2}{4}\int_0^\pi (\cos 3\xi + 3\cos\xi)\cos 3\xi \, d\xi$$

$$\Rightarrow c_2 = 0 \cdot c_1 + \frac{\lambda c_2}{4}\int_0^\pi \cos^2 3\xi \, d\xi + 0$$

$$\Rightarrow c_2 = \frac{\lambda}{4} c_2 \int_0^\pi \frac{1+\cos 6\xi}{2}\, d\xi$$

$$\Rightarrow c_2 = \frac{\lambda c_2}{8}\int_0^\pi d\xi + 0$$

$$\Rightarrow 0c_1 + \left(1 - \frac{\lambda\pi}{8}\right)c_2 = 0 \quad \text{—— (7)}$$

The $D(\lambda)$ for (6) & (7) is

$$D(\lambda) = \begin{vmatrix} 1-\frac{\lambda\pi}{4} & 0 \\ 0 & 1-\frac{\lambda\pi}{8} \end{vmatrix}$$

To find eigen values, we put $D(\lambda) = 0$

$$\Rightarrow \left(1-\frac{\lambda\pi}{4}\right)\left(1-\frac{\lambda\pi}{8}\right) = 0 \Rightarrow \lambda_1 = \frac{4}{\pi}, \lambda_2 = \frac{8}{\pi}$$

To find eigen values corresponding to eigen value $\lambda_1 = \frac{4}{\pi}$

The corresponding system is given by

$$\left(1-\frac{\lambda_1}{4}\pi\right)c_1 = 0 \quad \Rightarrow 0c_1 = 0 \text{ for } \lambda_1 = \frac{4}{\pi}$$

$$\left(1-\frac{\lambda_1}{8}\pi\right)c_2 = 0 \quad \Rightarrow \frac{1}{2}c_2 = 0 \text{ for } \lambda_1 = \frac{4}{\pi}$$

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$\Rightarrow c_2 = 0$, Take c_1 as arbitrary

$\therefore$ The Eigen function is given by

$$\phi_1(x) = \lambda c_1 \cos^2 x + \lambda c_2 \cos 3x$$

$$\Rightarrow \phi_1(x) = \lambda c_1 \cos^2 x$$

let $\lambda c_1 = 1$

$\Rightarrow \phi_1(x) = \cos^2 x$ which is an eigen function corresponding to eigen value $\lambda_1 = \frac{4}{\pi}$.

To find eigen function corresponding to eigen value $\lambda_2 = \frac{8}{\pi}$ we have corresponding system as

$$\left(1 - \frac{\lambda_2}{4}\pi\right) c_1 = 0 \quad \Rightarrow \left(1 - \frac{1}{4} \cdot \frac{8}{\pi} \pi\right) c_1 = 0$$

$$\Rightarrow -c_1 = 0 \Rightarrow c_1 = 0$$

$$\left(1 - \frac{\lambda_2}{8}\pi\right) c_2 = 0$$

$$\Rightarrow 0\, c_2 = 0 \Rightarrow c_2 \text{ is arbitrary}$$

put $c_1 = 0$ in equation (2), we have

$$\phi_2(x) = \lambda c_1 \cos^2 x + \lambda c_2 \cos 3x$$

$$\Rightarrow \phi_2(x) = \lambda c_2 \cos 3x$$

Take $\lambda c_2 = 1$

$\Rightarrow \phi_2(x) = \cos 3x$, which is an eigen function corresponding to eigen value $\lambda_2 = \frac{8}{\pi}$.

(2) Prove that the homogeneous integral equation

$$\phi(x) = \lambda \int_0^1 (3x-2)\,\xi\, \phi(\xi)\, d\xi,$$

has no eigen values and eigen function.

Solution. Given $\phi(x) = \lambda \int_0^1 (3x-2)\,\xi\,\phi(\xi)\,d\xi$ — (1)

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$$\Rightarrow \phi(x) = \lambda(3x-2)\,c_1 \quad \text{———} \quad (2)$$

where
$$c_1 = \int_0^1 \xi\,\phi(\xi)\,d\xi \quad \text{———} \quad (3)$$

From (2)
$$\phi(\xi) = \lambda c_1 (3\xi - 2) \quad \text{———} \quad (4)$$

Equ (3) and (4) $\Rightarrow$

$$c_1 = \int_0^1 \xi\,[\lambda c_1 (3\xi-2)]\,d\xi$$

$$c_1 = \lambda c_1 \int_0^1 (3\xi^2 - 2\xi)\,d\xi$$

$$c_1 = \lambda c_1 [1-1]$$

$$\Rightarrow c_1 = 0$$

put $c_1 = 0$ into (2), we have

$\phi(x) = 0$ which is not an eigen function

Since coefficient of λ is zero it means There exist no eigen value and hence there is no eigen function.

(3) Determine the eigen values and eigen function of The following
$$\phi(x) = \lambda\int_{-1}^{1} (5x\xi^3 + 4x^2\xi + 3x\xi)\,\phi(\xi)\,d\xi$$

Solution. Given
$$\phi(x) = \lambda\int_{-1}^{1} (5x\xi^3 + 4x^2\xi + 3x\xi)\,\phi(\xi)\,d\xi \quad \text{—} \quad (1)$$

$$\Rightarrow \phi(x) = 5x\lambda c_1 + (4x^2+3x)\,c_2\,\lambda \quad \text{———} \quad (2)$$

where
$$c_1 = \int_{-1}^{1} \xi^3\,\phi(\xi)\,d\xi \quad \text{———} \quad (3)$$

$$c_2 = \int_{-1}^{1} \xi\,\phi(\xi)\,d\xi \quad \text{———} \quad (4)$$

From (2), we have
$$\phi(\xi) = 5\lambda c_1 \xi + \lambda c_2 (4\xi^2 + 3\xi) \quad \text{———} \quad (5)$$

Equations (3) and (5) $\Rightarrow$

$$c_1 = \int_{-1}^{1} \xi^3 \phi(\xi)\, d\xi$$

$$= \int_{-1}^{1} \xi^3 \left[5\lambda c_1 \xi + \lambda c_2 (4\xi^2 + 3\xi)\right] d\xi$$

$$= 5\lambda c_1 \int_{-1}^{1} \xi^4 \, d\xi + \lambda c_2 \int_{-1}^{1} (4\xi^5 + 3\xi^4)\, d\xi$$

$$= 2\lambda c_1 + \lambda c_2 \left[0 + \frac{6}{5}\right]$$

$$c_1 = 2\lambda c_1 + \frac{6}{5}\lambda c_2$$

$$\Rightarrow (1-2\lambda)c_1 - \frac{6\lambda}{5} c_2 = 0 \qquad \text{——— (6)}$$

Equations (4) and (5) $\Rightarrow$

$$c_2 = \int_{-1}^{1} \xi \phi(\xi)\, d\xi$$

$$= \int_{-1}^{1} \xi \left[5\lambda c_1 \xi + \lambda c_2 (4\xi^2 + 3\xi)\right] d\xi$$

$$= 5\lambda c_1 \int_{-1}^{1} \xi^2 d\xi + 4\lambda c_2 \int_{-1}^{1} \xi^3 d\xi + 3\lambda c_2 \int_{-1}^{1} \xi^2 d\xi$$

$$c_2 = \frac{10\lambda c_1}{3} + 0 + 2\lambda c_2$$

$$\Rightarrow -\frac{10\lambda}{3} c_1 + (1-2\lambda) c_2 = 0 \qquad \text{——— (7)}$$

The characteristic equation of (6) and (7) $\Rightarrow$

$$D(\lambda) = \begin{vmatrix} 1-2\lambda & -\frac{6\lambda}{5} \\ -\frac{10\lambda}{3} & 1-2\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-2\lambda)^2 - \frac{60}{15}\lambda^2 = 0 \Rightarrow 1 + \cancel{4\lambda^2} - 4\lambda - \cancel{4\lambda^2} = 0$$

$$\Rightarrow \lambda = \frac{1}{4} = \lambda_1 \text{ say.}$$

To find eigen function corresponding to eigen value $\lambda_1 = \frac{1}{4}$.

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The corresponding system is given by

$$(1-2\lambda_1)c_1 - \frac{6\lambda_1}{5}c_2 = 0$$

$$-\frac{10\lambda_1}{3}c_1 + (1-2\lambda_1)c_2 = 0$$

put $\lambda_1 = \frac{1}{4}$

$$\left[1-2\left(\tfrac{1}{4}\right)\right]c_1 - \frac{6}{5}\left(\tfrac{1}{4}\right)c_2 = 0$$

$$\Rightarrow \quad \frac{c_1}{2} - \frac{3}{10}c_2 = 0 \Rightarrow 5c_1 - 3c_2 = 0 \quad \text{--- (8)}$$

$$\left[-\frac{10}{3}\left(\tfrac{1}{4}\right)\right]c_1 + \left[1-2\left(\tfrac{1}{4}\right)\right]c_2 = 0$$

$$-\frac{5}{6}c_1 + \frac{1}{2}c_2 = 0$$

$$-5c_1 + 3c_2 = 0 \quad \text{--- (9)}$$

$$\Rightarrow 5c_1 - 3c_2 = 0$$

Equ (8) & (9) $\Rightarrow$

$$c_1 = \frac{3}{5}c_2$$

put $c_1 = \frac{3}{5}c_2$ into (2) we have

$$\phi(x) = 5x\lambda_1 c_1 + (4x^2+3x)\lambda_1 c_2$$

$$= 5x\lambda_1\left(\frac{3}{5}c_2\right) + (4x^2+3x)\lambda_1 c_2$$

$$= \lambda_1 c_2\left[3x + 4x^2 + 3x\right]$$

$$\phi(x) = \lambda_1 c_2\left[4x^2+6x\right]$$

$$\phi_1(x) = \frac{1}{4}c_2(4x^2+6x)$$

$$\phi_1(x) = c_2\left[x^2+\frac{3}{2}x\right]$$

Take $c_2 = 1$

$\phi_1(x) = x^2 + \frac{3}{2}x$ which is required eigen fn. corresponding to eigen value $\lambda_1 = \frac{1}{4}$.

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(4) Find eigen values and eigen function of

$$\phi(x) = \lambda\int_0^1 (2x\xi - 4x^2)\,\phi(\xi)\,d\xi$$

Solution. Given

$$\phi(x) = \lambda\int_0^1 (2x\xi - 4x^2)\,\phi(\xi)\,d\xi \quad \text{—— (1)}$$

$$\Rightarrow \phi(x) = 2x\lambda c_1 - 4\lambda x^2 c_2 \quad \text{—— (2)}$$

where

$$c_1 = \int_0^1 \xi\,\phi(\xi)\,d\xi \quad \text{—— (3)}$$

$$c_2 = \int_0^1 \phi(\xi)\,d\xi \quad \text{—— (4)}$$

From (2), we have

$$\phi(\xi) = 2\lambda c_1\cdot\xi - 4\lambda c_2\,\xi^2 \quad \text{—— (5)}$$

Equations (3) and (5) $\Rightarrow$

$$c_1 = \int_0^1 \xi\left[2\lambda c_1\xi - 4\lambda c_2\xi^2\right]d\xi$$

$$\Rightarrow c_1 = 2\lambda c_1\int_0^1 \xi^2 d\xi - 4\lambda c_2\int_0^1 \xi^3 d\xi$$

$$c_1 = \frac{2}{3}\lambda c_1 - \lambda c_2$$

$$\Rightarrow \left(1 - \frac{2}{3}\lambda\right)c_1 + \lambda c_2 = 0 \quad \text{—— (6)}$$

Equations (4) and (5) $\Rightarrow$

$$c_2 = \int_0^1 (2\lambda c_1\xi - 4\lambda c_2\xi^2)\,d\xi$$

$$c_2 = 2\lambda c_1\int_0^1 \xi\,d\xi - 4\lambda c_2\int_0^1 \xi^2 d\xi$$

$$\Rightarrow c_2 = \lambda c_1 - \frac{4}{3}\lambda c_2$$

$$\Rightarrow -\lambda c_1 + \left(1 + \frac{4}{3}\lambda\right)c_2 = 0 \quad \text{—— (7)}$$

The characteristic equation for (6) & (7) is given by

$$D(\lambda) = \begin{vmatrix} 1 - \frac{2}{3}\lambda & \lambda \\ -\lambda & 1 + \frac{4}{3}\lambda \end{vmatrix} = 0$$

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$$\left(1-\frac{2}{3}\lambda\right)\left(1+\frac{4}{3}\lambda\right)+\lambda^2=0$$

$$\Rightarrow \frac{(3-2\lambda)(3+4\lambda)}{9}+\lambda^2=0$$

$$\Rightarrow 9+12\lambda-6\lambda-8\lambda^2+9\lambda^2=0$$

$$\Rightarrow 9+6\lambda+\lambda^2=0$$

$$\Rightarrow (3+\lambda)^2=0 \quad \Rightarrow \lambda=-3,-3=\lambda_1, \text{ say}$$

To find eigen vector (or eigenfunction) corresponding to eigen values $\lambda_1=\lambda_2=-3$

The corresponding system is given by

$$\left(1-\frac{2\lambda_1}{3}\right)c_1+\lambda_1 c_2=0$$

$$-\lambda_1+\left(1+\frac{4\lambda_1}{3}\right)c_2=0$$

$$\Rightarrow \left[1-\frac{2}{3}(-3)\right]c_1+(-3)c_2=0 \Rightarrow 3c_1-3c_2=0$$
$$\Rightarrow c_1=c_2 \quad \text{— (8)}$$

$$-(-3)c_1+\left[1+\frac{4}{3}(-3)\right]c_2=0$$

$$\Rightarrow 3c_1-3c_2=0 \quad \Rightarrow c_1=c_2 \quad \text{— (9)}$$

from (8) & (9), we have $c_1=c_2$

So put $c_1=c_2$ into (2), we have

$$\phi_1(x)=2x\lambda_1 c_1-4\lambda_1 c_2 x^2$$

$$\phi_1(x)=2x(-3)c_2-4(-3)c_2x^2$$

$$=-6c_2x+12c_2x^2$$

$$\phi_1(x)=-6c_2\left[x-2x^2\right]$$

Take $-6c_2=1$

$\therefore$ $\phi_1(x)=(x-2x^2)$ which is an eigen function corresponding to eigen values $\lambda_1=\lambda_2=-3$.

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(5) Solve

$$\phi(x) = \frac{1}{e^2-1}\int_0^1 2e^x e^\xi \phi(\xi)\,d\xi$$

Given $\phi(x) = \frac{1}{e^2-1}\int_0^1 2e^x e^\xi \phi(\xi)\,d\xi$ —— (1)

$\Rightarrow \phi(x) = \frac{1}{e^2-1}\cdot 2e^x \cdot c_1$ —— (2)

where

$c_1 = \int_0^1 e^\xi \phi(\xi)\,d\xi$ —— (3)

From (2), we get

$\phi(\xi) = \frac{1}{e^2-1}\, 2 \cdot e^\xi c_1$ —— (4)

Equations (3) and (4) $\Rightarrow$

$$c_1 = \int_0^1 e^\xi \left[\frac{1}{e^2-1}\, 2e^\xi c_1\right] d\xi$$

$$c_1 = \frac{2c_1}{e^2-1}\int_0^1 e^{2\xi}\,d\xi \Rightarrow c_1 = \frac{2c_1}{\cancel{(e^2-1)}}\cdot\cancel{(e^2-1)}$$

$\Rightarrow c_1 = 0$

$\therefore$ from (2), we have

$\phi(x) = \frac{1}{e^2-1}\cdot 2e^x \cdot 0 = 0$ which is not an eigen

$\Rightarrow$ function so there is no eigen value.

(6) Prove that the Integral equation

$$\phi(x) = \lambda\int_0^1 \left[\sqrt{x}\cdot\xi - \sqrt{\xi}\cdot x\right]\phi(\xi)\,d\xi$$

does not have real characteristic numbers and characteristic functions,

Solution. Given

$\phi(x) = \lambda\int_0^1 \left[\sqrt{x}\cdot\xi - \sqrt{\xi}\cdot x\right]\phi(\xi)\,d\xi$ —— (1)

$\Rightarrow \phi(x) = \lambda\sqrt{x}\,c_1 - \lambda x c_2$ —— (2)

where $c_1 = \int_0^1 \xi\,\phi(\xi)\,d\xi$ —— (3)

$c_2 = \int_0^1 \sqrt{\xi}\,\phi(\xi)\,d\xi$ —— (4)

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from (2), we have

$$\phi(\xi) = \lambda c_1 \sqrt{\xi} - \lambda c_2 \xi \qquad \text{—— (5)}$$

Equ's (3) and (5) ⇒

$$c_1 = \int_0^1 \xi \, \phi(\xi) d\xi = \int_0^1 \xi \left[\lambda c_1 \sqrt{\xi} - \lambda c_2 \xi \right] d\xi$$

$$\Rightarrow c_1 = \lambda c_1 \int_0^1 \xi^{3/2} d\xi - \lambda c_2 \int_0^1 \xi^2 d\xi$$

$$= \lambda c_1 \cdot \left[\frac{\xi^{5/2}}{5/2} \right]_0^1 - \lambda c_2 \cdot \frac{1}{3}$$

$$c_1 = \frac{2}{5} \lambda c_1 - \frac{\lambda}{3} c_2$$

$$\Rightarrow \left(1 - \frac{2}{5}\lambda\right) c_1 + \frac{\lambda}{3} c_2 = 0 \qquad \text{—— (6)}$$

Equations (4) and (5) ⇒

$$c_2 = \int_0^1 \sqrt{\xi} \, \phi(\xi) d\xi = \int_0^1 \sqrt{\xi} \left[\lambda c_1 \sqrt{\xi} - \lambda c_2 \xi \right] d\xi$$

$$= \lambda c_1 \int_0^1 \xi \, d\xi - \lambda c_2 \int_0^1 \xi^{3/2} d\xi$$

$$c_2 = \frac{\lambda c_1}{2} - \frac{2\lambda}{5} c_2$$

$$\Rightarrow -\frac{\lambda c_1}{2} + \left(1 + \frac{2\lambda}{5}\right) c_2 = 0 \qquad \text{—— (7)}$$

The characteristic polynomial for (6) & (7) is

$$D(\lambda) = \begin{vmatrix} 1 - \frac{2}{5}\lambda & \frac{\lambda}{3} \\ -\frac{\lambda}{2} & 1 + \frac{2\lambda}{5} \end{vmatrix} = 1 + \frac{\lambda^2}{150}$$

if we put $D(\lambda) = 0$ we can not get real value of λ.

$$\therefore c_1 = \frac{1}{D(\lambda)} \begin{vmatrix} 0 & \frac{\lambda}{3} \\ 0 & 1 + \frac{2\lambda}{5} \end{vmatrix} = 0$$

$$c_2 = \frac{1}{D(\lambda)} \begin{vmatrix} 1 - \frac{2}{5}\lambda & 0 \\ -\frac{\lambda}{2} & 0 \end{vmatrix} = 0$$

$\therefore \phi(x) = 0$, which is not an eigen function.

$\therefore$ Eq^n has no real eigen value and eigen function.

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